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EXTREME POINTS OF CONVEX SETS

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1. The starting point of the present paper is the simple observation that an element a of a convex set K is an extreme point of K if and only if $(K - a) \cap (K - a) = \{0\}$. Forming the polar set of each side of this equation we get criteria for a to be an extreme point of K. Theorems of this kind were first proved by R.C. Buck ((5)) and later proved in a more general setting by R.R. Phelps ((6)). Our method of proof is somewhat different, and also permits us to prove similar characterizations of the strict extreme points and of the exposed points of K (see the definitions following (3.1) and (3.2)). We give in section 5 a proof of the main result of Buck and Phelps. In the concluding section we show how some of our results are related to a recent theorem of Hervé ((7)).

In our setting we are given two real vector spaces E and F with a dual pairing $\langle \; , \; \rangle$. This notion is studied in ((2)) and we shall follow the terminology of this book. For instance, if $A \subset E$ then A^{o}, the *polar* set of A, is defined by $A^{o} = \{y \in F : \langle x,y \rangle \leq 1, \forall x \in A\}$. We also mention that the Mackey-topology in E is denoted $\tau(E,F)$. This is the strongest locally convex topology in E with F as the dual space. Unless otherwise stated, topological notions refer to the weak topologies in E and F.

2.1. *Lemma*. Let A and B be closed, convex subsets of E, both containing 0. Then

$$(2.2) \qquad A \cap B = \{0\}$$

if and only if $A^{o} + B^{o}$ is dense in F.

If, in addition, A is compact, then (2.2) is valid if and only if

$$(2.3) \qquad A^{o} + B^{o} = F .$$

Proof. Using the bi-polarity theorem (see e.g. ((2, p. 52))) we conclude that (2.2) is valid if and only if

$$(2.4) \qquad (A \cap B)^{o} = F .$$

It is furthermore known (loc. cit.) that $(A \cap B)^{o}$ is the closed convex hull of $A^{o} \cup B^{o}$. Hence (2.4) means that the convex hull of $A^{o} \cup B^{o}$ is a dense subset of F . Now we have

$$\tfrac{1}{2}(A^{o} + B^{o}) \subset \text{convex hull of } A^{o} \cup B^{o} \subset A^{o} + B^{o} .$$

Therefore the convex hull of $A^{o} \cup B^{o}$ is dense in F if and only if $A^{o} + B^{o}$ is dense in F . This proves the first part of the lemma.

If A is compact, then it follows from Mackey's theorem ((2, p. 68)) that A^{o} is a convex zero-neighbourhood in the Mackey-topology $\tau(F,E)$. It is then an elementary fact ((1, p. 51)) that the convex set $A^{o} + B^{o}$ is a τ-dense subset of F if and only if $A^{o} + B^{o} = F$. Since a convex set in F is τ-dense if and only if it is weakly dense, the proof is finished.

3. Let K be a closed, convex subset of E , and let $a \in K$. As usual, a is called an extreme point of K if a is not the midpoint of a line segment contained in K . This means that

$$(3.1) \qquad (K - a) \cap (a - K) = \{0\} .$$

We say that a is an e x p o s e d point of K if there exists a y in F such that

$$\langle x,y \rangle > \langle a,y \rangle , \quad \forall\, x \in K \setminus \{a\} .$$

Thus a is an exposed point if and only if for some y in F ,

$$(3.2) \qquad (K - a) \cap \{ x : \langle x,y \rangle \leq 0 \} = \{0\} .$$

Finally, a is called a s t r i c t extreme point, a property between that of being extreme and exposed, if for any $x_o \in K \setminus \{a\}$, there exists a y in F such that

$$(3.3) \qquad \langle a,y\rangle = \inf\{\langle x,y\rangle : x \in K\} < \langle x_o,y\rangle \quad .$$

Using the separation theorem for convex sets, we get that (3.3) is equivalent with

$$a - x_o \notin \overline{\bigcup_{\lambda>0} \lambda(K - a)}$$

Hence a is a strict extreme point of K if and only if

$$(3.4) \qquad (a - K) \cap \overline{\bigcup_{\lambda>0} \lambda(K - a)} = \{0\} \quad .$$

We are now ready to apply Lemma 2.1. We could state a more general theorem, assuming only that K is closed, but we prefer to restrict ourselves to the case where K is compact.

3.5. Theorem. Assume that K is compact and let $a \in K$. Then

(i) a is an extreme point of K if and only if

$$(3.6) \qquad (K - a)^o - (K - a)^o = F \quad .$$

(ii) a is a strict extreme point of K if and only if

$$(3.7) \qquad (K - a)^o - \bigcap_{\lambda>0} \lambda(K - a)^o = F \quad .$$

(iii) a is an exposed point of K if and only if there exists a $y \in F$ such that

$$(3.8) \qquad (K - a)^o + \{\lambda y : \lambda \geq 0\} = F \quad .$$

Proof. Applying Lemma 2.1 and (3.1), (3.2) and (3.4) the only thing to prove is that

$$(3.9)\qquad \overline{\Big(\bigcup_{\lambda>0}\lambda(K-a)\Big)}^{o} = \bigcap_{\lambda>0}\lambda(K-a)^{o}$$

and that

$$(3.10)\qquad \{x : \langle x,y\rangle \leq 0\}^{o} = \{\lambda y : \lambda \geq 0\} .$$

(3.9) expresses a well known fact about polar sets. To prove (3.10) we have only to observe that if $z \in F$ has the property that $\langle x,y\rangle \leq 0$ implies $\langle x,z\rangle \leq 1$, then $y^{-1}(0) \subset z^{-1}(0)$ and therefore $z = \lambda y$ with $\lambda \geq 0$.

4. In this section we assume that P is a convex cone in F , with a non-empty interior with respect to $\tau(F,E)$ and with zero as a vertex. Furthermore, let $e \in \mathrm{int}.P$. Define

$$(4.1)\qquad Q = \{x \in E : -x \in P^{o} \;\&\; \langle x,e\rangle = 1\} .$$

Hence $x \in E$ belongs to Q if and only if

$$(4.2)\qquad \langle x,e\rangle = 1 \quad \text{and} \quad \langle x,y\rangle \geq 0 ,\ \forall y \in P .$$

It is known ((3)) that Q is compact.

4.3. L e m m a . Let $x_o \in Q$. Then

$$(Q - x_o)^{o} = \{\lambda e : \lambda \in R\} - \overline{P} \cap x_o^{-1}(1) .$$

P r o o f . Let $y \in (Q - x_o)^{o}$, and put

$$p = (\langle x_o,y\rangle + 1)e - y .$$

Then we have $\langle x_o, p\rangle = 1$. I claim that $p \in \overline{P}$, and to prove this, it is, according to the bi-polarity theorem, sufficient to prove that $\langle x,p\rangle \geqslant 0$ whenever $x \in Q$. Choose $x \in Q$. Then

$$\langle x,p\rangle = \langle x_o,y\rangle + 1 - \langle x,y\rangle = 1 - \langle x - x_o,y\rangle \geqslant 0 \quad ,$$

as desired. The rest of the proof is readily finished.

Somewhat misleading we shall say that an element of Q is a *positive normalized* linear functional (with respect to P) , and we shall call an extreme point of Q for an *extreme* positive linear functional.

4.4. *Theorem.* Let x be a positive normalized linear functional. Then x is an extreme positive linear functional if and only if

$$(4.5) \qquad F = \{\lambda e : \lambda \in R\} + P \cap x^{-1}(1) - P \cap x^{-1}(1) \quad .$$

Proof. Applying Lemma 4.3 and Theorem 3.5 (i), we have only to show that if

$$F = \{\lambda e : \lambda \in R\} + \overline{P} \cap x^{-1}(1) - \overline{P} \cap x^{-1}(1) \quad ,$$

then (4.5) is valid. Let $y \in F$ be given. Then

$$2y = \lambda e + p - q \quad , \qquad \text{where} \qquad p,q \in \overline{P} \cap x^{-1}(1) \quad .$$

Since $e \in \text{int.}P$, it is a well known fact ((1, p. 51)) that $\frac{1}{2}(e + p)$, $\frac{1}{2}(e + q) \in P$. Hence

$$y = \frac{\lambda}{2} e + \tfrac{1}{2}(e + p) - \tfrac{1}{2}(e + q)$$

belongs to the right hand side of (4.5) .

4.6. *Remark.* We infer from the Krein-Milman theorem that there always exists an x in E such that (4.5) is satisfied.

4.7. Example. We now assume that P gives rise to a lattice ordering in F. It is then known that a positive, normalized linear functional x is extreme if and only if $x^{-1}(0)$ is a lattice ideal, which means that $y \in x^{-1}(0)$ implies $|y| \in x^{-1}(0)$ (see e.g. ((4))). This result follows from Theorem 4.4 as follows: Assume that x is extreme and choose $y \in x^{-1}(0)$. Then we have $y = p - q$, with $p,q \in P \cap x^{-1}(1)$. Furthermore $y = y^+ - y^-$, and since $p,q \in P$ we get $y^+ \leq p$, $y^- \leq q$. Therefore $0 \leq \langle x,y^+\rangle \leq \langle x,p\rangle = 1$ and $0 \leq \langle x,y^-\rangle \leq 1$. From this we conclude that $0 \leq \langle x,y^+ + y^-\rangle = \langle x, |y|\rangle \leq 2$. Choosing λy instead of y with λ real we get $0 \leq \langle x, |\lambda y|\rangle = |\lambda| \langle x, |y|\rangle \leq 2$, and therefore $\langle x, |y|\rangle = 0$. Assume conversely that $x^{-1}(0)$ is a lattice-ideal. Let $y_1 \in F$ be given. Then $y = y_1 - \langle x,y_1\rangle e \in x^{-1}(0)$. Hence $|y| \in x^{-1}(0)$ and also $y^+, y^- \in x^{-1}(0)$. Therefore $y^+ + e$, $y^- + e$ $P \cap x^{-1}(1)$. Since

$$y_1 = \langle x,y_1\rangle e + y = \langle x,y_1\rangle e + (y^+ + e) - (y^- + e) \quad ,$$

we have proved (4.5), and so x is extreme.

5. In this section we establish the connection with the results of Buck and Phelps.

Let C be a convex zero-neighbourhood with respect to $\tau(F,E)$. Then C^0 is weakly compact ((2, p. 65)). Hence we can apply Theorem 3.5. For this purpose we are going to characterize the set $(C^0 - x)^0$. In order to do this we need to introduce the gauge-function p_C of C, defined as

$$p_C(y) = \inf\{\lambda > 0 : \lambda^{-1} y \in C\} \quad .$$

Since 0 is an interior point of C, we have

$$(5.1) \qquad p_C = p_{\overline{C}}$$

This follows from the fact that if $y \in \overline{C}$ and $0 < \lambda < 1$, then $\lambda y \in C$. We also need the support-function h_M of a bounded subset M in E. We define

$$h_M(y) = \sup.\{ \langle x,y \rangle : x \in M \} .$$

The fundamental result connecting these two concepts is the following equality ((2, p. 58 Exc. 5))

$$(5.2) \qquad p_{\overline{C}} = h_{C^o} .$$

5.3. Definition. For λ real and x in E we put

$$C(x,\lambda) = \{ y : p_{\overline{C}}(y) \leq \lambda + \langle x,y \rangle \} .$$

5.4. Lemma. Let $x \in E$. Then

$$(5.5) \qquad (C^o - x)^o = C(x,1)$$

Proof. Applying (5.1) and (5.2) we get that y belongs to $C(x,1)$ if and only if

$$\sup \{ \langle z,y \rangle : z \in C^o \} \leq 1 + \langle x,y \rangle .$$

This means that

$$\langle z - x,y \rangle \leq 1 , \forall z \in C^o ,$$

and, by definition, this is equivalent with $y \in (C^o - x)^o$.

Referring to Theorem 3.5 and Lemma 5.4 we obtain

5.6. Theorem. (R.C. Buck and R.R. Phelps.) Let C be a convex zero-neighbourhood with respect to $\tau(F,E)$. Then $x \in C^o$ is an extreme point of C^o if and only if

$$F = C(x,1) - C(x,1) \quad . \tag{5.7}$$

We can in a similar way characterize the strict extreme points and the exposed points of C^o.

5.8. Theorem. Let C be a convex zero-neighbourhood with respect to $\tau(F,E)$, and let $x \in C^o$. Then

(i) x is a strict extreme point of C^o if and only if

$$F = C(x,1) - C(x,0) \quad .$$

(ii) x is an exposed point of C^o if and only if there exists $y_o \in F$ such that

$$F = C(x,1) + \{\lambda y_o : \lambda \geq 0\} \quad .$$

Proof. Applying Theorem 3.5, the only thing we have to prove is that

$$\bigcap_{\lambda > 0} \lambda (C^o - x)^o = C(x,0) \quad ,$$

and this equality follows readily from Lemma 5.4.

6. Application. We now want to show how Theorem 4.4 is related to a theorem of Hervé ((7)). We assume that F is a linear subspace of $C(T)$, the space of all real continuous functions on a compact

space T. We also assume that F contains the identity function e. Furthermore, P is the set of all non-negative functions in F, and E is the dual space of F when F is equipped with the uniform norm topology.

We recall that a *linear sublattice* of $C(T)$ is a linear subspace L such that $l \in L$ implies that $|l| \in L$.

6.1. *Theorem*. Let $t \in T$ and let V be the closed linear sublattice generated by F. Then the positive functional

$$\hat{t} : y \to y(t) , \quad \forall \; y \in F$$

is an extreme functional if and only if

$$(6.2) \qquad g(t) = \inf \{ p(t) : p \in F \;\&\; p \geq g \}, \; \forall \; g \in V .$$

Proof. It follows immediately from Theorem 4.4 that the condition (6.2) is sufficient. To prove that it is necessary, we define G as the set of all g in $C(T)$ such that

$$0 = \inf \{ p(t) : p \in F \;\&\; p \geq | g - g(t)e | \} .$$

We are through if we can prove that V is contained in G. It is plain that G is a linear space, and since

$$\big| |g| - |g(t)| \, e \big| \leq | g - g(t)e | ,$$

we can conclude that G is a linear lattice. We claim that G is closed under uniform convergence. Indeed, let $g_n \to g_o$, where $g_n \in G$. Let $\varepsilon > 0$ be given. We can find $p_n \in F$ such that $|g_n - g_n(t)e| \leq p_n$ and such that $p_n(t) \leq \frac{\varepsilon}{3}$. Choose n such that $|g_n - g_o| \leq \frac{\varepsilon}{3} e$. We then get $| g_o - g_o(t) \cdot e | \leq \frac{2}{3} \varepsilon e + p_n$. Since $\frac{2}{3} \varepsilon e + p_n \in F$ and $\frac{2}{3} \varepsilon + p_n(t) \leq \varepsilon$, we infere that $g_o \in G$. Finally, F is contained in

G . Because, let $y \in F$ be given. According to Theorem 4.4, we can find, for any $\varepsilon > 0$, elements p, q in P such that

$$\varepsilon^{-1} y - \varepsilon^{-1} ey(t) = p - q , \text{ and } p(t) = q(t) = 1 .$$

Therefore

$$|y - y(t)e| \leq \varepsilon(p + q) \in F \quad \text{and} \quad \varepsilon(p + q)(t) = 2\varepsilon .$$

Hence $y \in G$. Therefore $V \subset G$, as was to be proved.

Using Stone's approximation theorem for a linear sublattice of $C(T)$, we get

6.3. C o r o l l a r y . (M. Hervé). Assume that F separates the points of T . Then $\hat{t}$ is an extreme functional if and only if

$$g(t) = \inf \{p(t) : p \in F \ \& \ p \geq g\} , \forall g \in C(T) .$$

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